

# CHARACTERIZATION OF SIMPLICES VIA THE BEZOUT INEQUALITY FOR MIXED VOLUMES

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ABSTRACT. We consider the following Bezout inequality for mixed volumes:

$$V(K_1, \dots, K_r, \Delta[n-r])V_n(\Delta)^{r-1} \leq \prod_{i=1}^r V(K_i, \Delta[n-1]) \quad \text{for } 2 \leq r \leq n.$$

It was shown previously that the inequality is true for any  $n$ -dimensional simplex  $\Delta$  and any convex bodies  $K_1, \dots, K_r$  in  $\mathbb{R}^n$ . It was conjectured that simplices are the only convex bodies for which the inequality holds for arbitrary bodies  $K_1, \dots, K_r$  in  $\mathbb{R}^n$ . In this paper we prove that this is indeed the case if we assume that  $\Delta$  is a convex polytope. Thus the Bezout inequality characterizes simplices in the class of convex  $n$ -polytopes. In addition, we show that if a body  $\Delta$  satisfies the Bezout inequality for all bodies  $K_1, \dots, K_r$  then the boundary of  $\Delta$  cannot have *strict* points. In particular, it cannot have points with positive Gaussian curvature.

## 1. INTRODUCTION

It was noticed in [SZ] that the classical Bezout inequality in algebraic geometry [F, Sec. 8.4] together with the Bernstein–Kushnirenko–Khovanskii bound [B, Ku, Kh] produces a new inequality involving mixed volumes of convex bodies:

$$(1.1) \quad V(K_1, \dots, K_r, \Delta[n-r])V_n(\Delta)^{r-1} \leq \prod_{i=1}^r V(K_i, \Delta[n-1]) \quad \text{for } 2 \leq r \leq n.$$

Here  $\Delta$  is an  $n$ -dimensional simplex and  $K_1, \dots, K_r$  are arbitrary convex bodies in  $\mathbb{R}^n$ . Throughout the paper  $V_n(K)$  denotes the  $n$ -dimensional Euclidean volume of a body  $K$  and  $V(K_1, \dots, K_n)$  denotes the  $n$ -dimensional mixed volume of bodies  $K_1, \dots, K_n$ . Furthermore,  $K[m]$  indicates that the body  $K$  is repeated  $m$  times in the expression for the mixed volume.

In [SZ] it was conjectured that the Bezout inequality characterizes simplices, that is if  $\Delta$  is a convex body such that (1.1) holds for all convex bodies  $K_1, \dots, K_r$  then  $\Delta$  is necessarily a simplex (see [SZ, Conjecture 1.2]). It was proved that  $\Delta$  has to be indecomposable (see [SZ, Theorem 3.3]) which, in particular, confirms the conjecture in dimension  $n = 2$ . In the present paper we prove this conjecture for the class of convex polytopes.

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2010 *Mathematics Subject Classification*. Primary 52A39, 52A40, 52B11.

*Key words and phrases*. Convex Bodies, Mixed Volume, Convex Polytopes, Bezout Inequality, Aleksandrov–Fenchel Inequality.

The third author is supported in part by U.S. National Science Foundation Grant DMS-1101636 and by the Simons Foundation.

**Theorem 1.1.** *Fix  $2 \leq r \leq n$ . Let  $\Delta$  be a convex  $n$ -dimensional polytope in  $\mathbb{R}^n$  satisfying (1.1) for all convex bodies  $K_1, \dots, K_r$  in  $\mathbb{R}^n$ . Then  $\Delta$  is a simplex.*

Although the above theorem covers a most natural class of convex bodies, in full generality the conjecture remains open. Going outside of the class of polytopes we show that if a convex body  $\Delta$  satisfies (1.1) for all convex bodies  $K_1, \dots, K_r$  in  $\mathbb{R}^n$  then  $\Delta$  cannot have strict points. We say a boundary point  $x \in K$  is a *strict point* if  $x$  does not belong to any segment contained in the boundary of  $K$ .

**Theorem 1.2.** *Fix  $2 \leq r \leq n$ . Let  $\Delta$  be an  $n$ -dimensional convex body in  $\mathbb{R}^n$  satisfying (1.1) for all convex bodies  $K_1, \dots, K_r$  in  $\mathbb{R}^n$ . Then  $\Delta$  does not contain any strict points.*

In particular, we see that  $\Delta$  cannot have points with positive Gaussian curvature.

Let us say a few words about the idea behind the proofs of Theorems 1.1 and 1.2. First, note that it is enough to prove the theorems in the case of  $r = 2$  as this implies the general statement. Thus we are going to restate (1.1) for  $r = 2$  as follows

$$(1.2) \quad V(L, M, K[n-2])V_n(K) \leq V(L, K[n-1])V(M, K[n-1]),$$

where  $L$  and  $M$  are convex bodies and  $K$  is a polytope. The fact that there is equality in (1.2) when  $L = K$  allows us to see this as a variational problem, by fixing an appropriate body  $M$  and using an appropriate deformation  $L = K_t$  of  $K$ . In the case of Theorem 1.1,  $K_t$  is obtained from  $K$  by moving one of its facets along the direction of its normal unit vector. In the case of Theorem 1.2,  $K_t$  is obtained from  $K$  by cutting out a small cup in a neighborhood of a strict point.

## 2. PRELIMINARIES

In this section we collect basic definitions and set up notation. As a general reference on the theory of convex sets and mixed volumes we use R. Schneider's book "Convex bodies: the Brunn-Minkowski theory" [Sch].

A *convex body* is a non-empty convex compact set. A (*convex*) *polytope* is the convex hull of a finite set of points. An  $n$ -dimensional polytope is called an  $n$ -*polytope* for short. For  $x, y \in \mathbb{R}^n$  we write  $\langle x, y \rangle$  for the inner product of  $x$  and  $y$ . We use  $\mathbb{S}^{n-1}$  to denote the  $(n-1)$ -dimensional unit sphere and  $B(x, \delta)$  to denote the closed Euclidean ball of radius  $\delta > 0$  centered at  $x \in \mathbb{R}^n$ .

For a convex body  $K$  the function  $h_K : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ ,  $h_K(u) = \max\{\langle x, u \rangle \mid x \in K\}$  is the *support function* of  $K$ . For every  $u \in \mathbb{S}^{n-1}$  we write  $H_K(u)$  to denote the supporting hyperplane for  $K$  with outer normal  $u$

$$H_K(u) = \{x \in \mathbb{R}^n : \langle x, u \rangle = h_K(u)\}.$$

Furthermore, we use  $K^u$  to denote the face  $K \cap H_K(u)$  of  $K$ .

Let  $\beta$  be a subset of the boundary  $\partial K$  of a convex body  $K$ . The *spherical image*  $\sigma(K, \beta)$  of  $\beta$  with respect to  $K$  is defined by

$$\sigma(K, \beta) = \{u \in \mathbb{S}^{n-1} : \exists x \in \beta, \text{ such that } \langle x, u \rangle = h_K(u)\}.$$

If  $\Omega$  is a subset of  $\mathbb{S}^{n-1}$  define the *inverse spherical image*  $\tau(K, \Omega)$  of  $\Omega$  with respect to  $K$  by

$$\tau(K, \Omega) = \{x \in \partial K : \exists u \in \Omega, \text{ such that } \langle x, u \rangle = h_K(u)\}.$$

The *surface area measure*  $S(K, \cdot)$  of  $K$  (viewed as a measure on  $\mathbb{S}^{n-1}$ ) is defined as

$$S(K, \Omega) = \mathcal{H}^{n-1}(\tau(K, \Omega)), \quad \text{for } \Omega \text{ a Borel subset of } \mathbb{S}^{n-1}.$$

Here  $\mathcal{H}^{n-1}(\cdot)$  stands for the  $(n-1)$ -dimensional Hausdorff measure.

Let  $V(K_1, \dots, K_n)$  denote the  $n$ -dimensional mixed volume of  $n$  convex bodies  $K_1, \dots, K_n$  in  $\mathbb{R}^n$ . We write  $V(K_1[m_1], \dots, K_r[m_r])$  for the mixed volume of the bodies  $K_1, \dots, K_r$  where each  $K_i$  is repeated  $m_i$  times and  $m_1 + \dots + m_r = n$ . In particular,  $V(K[n]) = V_n(K)$ , the  $n$ -dimensional Euclidean volume of  $K$ .

Let  $S(K_1, \dots, K_{n-1}, \cdot)$  be the *mixed area measure* for bodies  $K_1, \dots, K_{n-1}$  defined by

$$V(L, K_1, \dots, K_{n-1}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L dS(K_1, \dots, K_{n-1}, \cdot)$$

for any compact convex set  $L$ . In particular, when the  $K_i$  are polytopes the mixed area measure  $S(K_1, \dots, K_{n-1}, \cdot)$  has finite support and for every  $u \in \mathbb{S}^{n-1}$  we have

$$(2.1) \quad S(K_1, \dots, K_{n-1}, u) = V(K_1^u, \dots, K_{n-1}^u),$$

where  $V(K_1^u, \dots, K_{n-1}^u)$  is the  $(n-1)$ -dimensional mixed volume of the faces  $K_i^u$  translated to the subspace orthogonal to  $u$ , see [Sch, Sec 5.1].

Finally, for  $u \in \mathbb{S}^{n-1}$  the orthogonal projection of a set  $A \subset \mathbb{R}^n$  onto the subspace  $u^\perp$  orthogonal to  $u$  is denoted by  $A|u^\perp$ .

### 3. PROOF OF THEOREM 1.1

In this section we give a proof of Theorem 1.1. As mentioned in the introduction, it is enough to prove it for  $r = 2$  in which case we write the Bezout inequality as

$$(3.1) \quad V(L, M, K[n-2])V_n(K) \leq V(L, K[n-1])V(M, K[n-1]).$$

We assume that  $L, M$  are arbitrary convex bodies and  $K$  is a polytope in  $\mathbb{R}^n$ .

We need to set up additional notation. Let  $K$  be defined by inequalities

$$K = \bigcap_{j=1}^N \{x \in \mathbb{R}^n : \langle x, u_j \rangle \leq h_K(u_j)\},$$

where  $u_j$  are the outer normals to the facets of  $K$  (in some fixed order) and  $N$  is the number of facets of  $K$ . Denote by  $K_{t,i}$  the polytope obtained by moving the  $i$ -th facet of  $K$  by  $t$ , that is

$$K_{t,i} = \bigcap_{\substack{j=1 \\ j \neq i}}^N \{x \in \mathbb{R}^n : \langle x, u_j \rangle \leq h_K(u_j)\} \cap \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq h_K(u_i) + t\}.$$

By abuse of notation we let  $K_t$  denote  $K_{t,N}$ .

**Lemma 3.1.** *Let  $K$  and  $K_t$  be as above. Then there exists  $\delta = \delta(K)$  such that the following supports are equal*

$$\text{supp } S(K_t[r], K[n-1-r], \cdot) = \text{supp } S(K, \cdot)$$

for any  $0 \leq r \leq n-1$  and any  $t \in (-\delta, \delta)$ .

*Proof.* By (2.1) it is enough to show that  $V(K_t^u[r], K^u[n-1-r]) = 0$  if and only if  $V_{n-1}(K^u) = 0$ , that is  $K^u$  is not a facet of  $K$ . Indeed, by choosing  $\delta$  small enough we can ensure that  $K_t$  has the same facet normals as  $K$  and so  $\dim K_t^u = n-1$  whenever  $K^u$  is a facet of  $K$ . In this case  $V(K_t^u[r], K^u[n-1-r]) > 0$ .

Conversely, assume  $K^u$  is a face of  $K$  of dimension less than  $n-1$ . As before, for small enough  $t$  the face  $K_t^u$  also has dimension less than  $n-1$ . First, suppose  $K^u$  is not contained in the moving facet  $F = K \cap H_K(u_N)$ . Then  $h_K(u) = h_{K_t}(u)$  and so  $K^u \subseteq K_t^u$  for  $t \geq 0$  and  $K^u \supseteq K_t^u$  for  $t < 0$ . Then, by the monotonicity of the mixed volume, if  $t \geq 0$  then

$$0 \leq V(K_t^u[r], K^u[n-1-r]) \leq V_{n-1}(K_t^u) = 0,$$

and so  $V(K_t^u[r], K^u[n-1-r]) = 0$ . The case  $t < 0$  is similar.

Now suppose  $K^u$  is contained in the moving facet  $F$ . Then  $K^u \subseteq H_K(u) \cap H_K(u_N)$  and  $K_t^u \subseteq H_{K_t}(u) \cap H_{K_t}(u_N)$ . This shows that  $K^u$  and  $K_t^u$  are contained in two affine  $(n-2)$ -dimensional subspaces which are translates of the same linear subspace of dimension  $n-2$ . Therefore, for any collection of line segments  $(L_1, \dots, L_{n-1})$ , where  $L_i \subset K_t^u$  for  $1 \leq i \leq r$  and  $L_i \subset K^u$  for  $r+1 \leq i \leq n-1$ , the  $L_i$  have linearly dependent directions. The latter implies that  $V(K_t^u[r], K^u[n-1-r]) = 0$  by [Sch, Theorem 5.1.7]. □

**Proposition 3.2.** *Let  $K, P$  be  $n$ -polytopes with the following properties:*

- (1)  $\text{supp } S(P, \cdot) = \text{supp } S(K, \cdot)$ ,
- (2) *there exists a constant  $\lambda > 0$  such that  $V(L, P[n-1]) \leq \lambda V(L, K[n-1])$  for all convex bodies  $L$ ,*
- (3)  $V(K, P[n-1]) = \lambda V_n(K)$ .

*Then,*

$$S(P, \cdot) = \lambda S(K, \cdot).$$

*Proof.* As before, let  $\{u_1, \dots, u_N\}$  be the outer normals to the facets of  $K$ . By assumption (1) they are the outer normals to the facets of  $P$  as well. Fix  $1 \leq i \leq N$  and let  $L = K_{s,i}$  be the polytope obtained from  $K$  by moving its  $i$ -th facet by a small number  $s \in (-\delta_i, \delta_i)$  as in Lemma 3.1.

By assumption (2), for any  $s \in (-\delta_i, \delta_i)$  we have

$$V(K_{s,i}, P[n-1]) \leq \lambda V(K_{s,i}, K[n-1]).$$

Consider the function

$$F(s) = \lambda V(K_{s,i}, K[n-1]) - V(K_{s,i}, P[n-1]).$$

Then  $F(s) \geq 0$  and  $F(0) = 0$ . Below we show that  $F(s)$  is, in fact, linear on  $(-\delta_i, \delta_i)$ . But then  $F(s)$  is identically zero on  $(-\delta_i, \delta_i)$ , which implies that

$$(3.2) \quad V(K_{s,i}, P[n-1]) = \lambda V(K_{s,i}, K[n-1])$$

for all  $s \in (-\delta_i, \delta_i)$ . We claim that this also implies that

$$(3.3) \quad S(P, u_i) = \lambda S(K, u_i),$$

and since  $i$  is chosen arbitrarily and the supports of the two measures are equal, the statement of the proposition follows.

Now we show that  $F(s)$  is linear and then prove that (3.2) implies (3.3). Since the polytopes  $P$  and  $K$  have the same set of facet normals  $\{u_1, \dots, u_N\}$ , we obtain:

$$\begin{aligned}
 nV(K_{s,i}, P[n-1]) &= \sum_{j=1}^N h_{K_{s,i}}(u_j) V_{n-1}(P^{u_j}) \\
 &= \sum_{j=1}^N h_K(u_j) V_{n-1}(P^{u_j}) + (h_K(u_i) + s) V_{n-1}(P^{u_i}) \\
 &= nV(K, P[n-1]) + sV_{n-1}(P^{u_i}) \\
 (3.4) \qquad &= n\lambda V_n(K) + sV_{n-1}(P^{u_i}).
 \end{aligned}$$

Similarly,

$$(3.5) \qquad nV(K_{s,i}, K[n-1]) = nV_n(K) + sV_{n-1}(K^{u_i}).$$

Substituting (3.4) and (3.5) into the definition of  $F(s)$  and using assumption (3), we see that  $F(s) = \lambda s$  for some  $\lambda$ , that is  $F(s)$  is linear.

It remains to show that (3.2) implies (3.3). Since  $F(s)$  is identically zero we have  $\lambda = 0$ , which translates to

$$V_{n-1}(P^{u_i}) = \lambda V_{n-1}(K^{u_i}).$$

But that is precisely what (3.3) is stating, which completes the proof of the proposition.  $\square$

**Lemma 3.3.** *Let  $K$  be an  $n$ -polytope satisfying (3.1) for all bodies  $L$  and for all  $M = K_t$  where  $t \in (-\delta, \delta)$  as in Lemma 3.1. Then*

$$S(K_t[r], K[n-1-r], \cdot) = \frac{V(K_t, K[n-1])^r}{V_n(K)^r} S(K, \cdot)$$

for all  $0 \leq r \leq n-1$  and all  $t \in (-\delta, \delta)$ .

*Proof.* For  $0 \leq r \leq n-1$ , set  $P_r$  to be the polytope whose surface area measure equals  $S(K_t[r], K[n-1-r], \cdot)$  and let  $\lambda := V(K_t, K[n-1])/V_n(K)$ . For each  $r$  the existence and uniqueness of  $P_r$  is ensured by the Minkowski Existence and Uniqueness Theorem (see [Sch, Sections 7.1, 7.2]). We need to prove that

$$(3.6) \qquad S(P_r, \cdot) = \lambda^r S(K, \cdot), \qquad r = 0, 1, \dots, n-1.$$

Note that by Lemma 3.1, we have:

$$(3.7) \qquad \text{supp } S(P_r, \cdot) = \text{supp } S(K, \cdot), \qquad r = 1, \dots, n-1.$$

We prove (3.6) by induction on  $r$ . The case  $r = 0$  is trivial. For the case  $r = 1$  we apply Proposition 3.2 with  $P = P_1$ . Indeed, by our assumption, (3.1) is satisfied for  $M = K_t$  and becomes equality when  $L = K$ . Thus the conditions (1)–(3) of Proposition 3.2 hold and so  $S(P_1, \cdot) = \lambda S(K, \cdot)$ , as required.

Now assume (3.6) holds for  $1 \leq m \leq r-1$ . This is equivalent to the following:

$$(3.8) \qquad V(L, P_m[n-1]) = \lambda^m V(L, K[n-1]),$$

for all convex bodies  $L$  and  $1 \leq m \leq r-1$ . Next fix a convex body  $L \subset \mathbb{R}^n$  and apply the Aleksandrov-Fenchel inequality

$$\begin{aligned}
& V(L, P_{r-1}[n-1])^2 = V(L, K_t[r-1], K[n-r])^2 \\
& = V(K, K_t, K_t[r-2], K[n-r-1], L)^2 \\
& \geq V(K, K, K_t[r-2], K[n-r-1], L) V(K_t, K_t, K_t[r-2], K[n-r-1], L) \\
& = V(L, K_t[r-2], K[n-r+1]) V(L, K_t[r], K[n-r-1]) \\
& = V(L, P_{r-2}[n-1]) V(L, P_r[n-1]) ,
\end{aligned}$$

which, by (3.8) with  $m = r-2$  and  $m = r-1$ , gives

$$\lambda^{2(r-1)} V(L, K[n-1])^2 \geq \lambda^{r-2} V(L, K[n-1]) V(L, P_r[n-1]).$$

Thus

$$(3.9) \quad V(L, P_r[n-1]) \leq \lambda^r V(K, P_r[n-1]).$$

Furthermore, using (3.8) for  $m = r-1$ , we get:

$$\begin{aligned}
V(K, P_r[n-1]) &= V(K, K_t[r], K[n-1-r]) \\
&= V(K_t, K_t[r-1], K[n-r]) \\
&= V(K_t, P_{r-1}[n-1]) \\
&= \lambda^{r-1} V(K_t, K[n-1]) \\
(3.10) \quad &= \frac{V(K_t, K[n-1])^{r-1}}{V_n(K)^{r-1}} V(K_t, K[n-1]) = \lambda^r V_n(K).
\end{aligned}$$

Now, as in the case of  $r=1$ , (3.7), (3.9), (3.10) together with Proposition 3.2, show that  $S(P_r, \cdot) = \lambda^r S(K, \cdot)$ , which completes the proof of the lemma.  $\square$

Now we are ready to prove the main theorem which implies Theorem 1.1.

**Theorem 3.4.** *Let  $K$  be an  $n$ -polytope in  $\mathbb{R}^n$ . Suppose that*

$$(3.11) \quad V(L, M, K[n-2]) V_n(K) \leq V(L, K[n-1]) V(M, K[n-1])$$

*holds for all convex bodies  $L$  and  $M$  in  $\mathbb{R}^n$ . Then  $K$  is a simplex.*

*Proof.* Let  $K_t$  be the polytope obtained by moving one of the facets of  $K$  for  $t$  small enough. Then Lemma 3.3 with  $r = n-1$  implies that the surface area measures of  $K_t$  and  $K$  are proportional, and hence,  $K_t$  is homothetic to  $K$ .

We may assume that one of the vertices of  $K$  not lying on the moving facet is at the origin, so  $K_t = \lambda K$  for some  $\lambda \neq 1$ . For every vertex  $v$  in  $K$ ,  $\lambda v$  must be a vertex of  $\lambda K$ . Therefore, the origin is the only vertex of  $K$  not lying on the moving facet. In other words,  $K$  is the cone over the moving facet. But since the facet was chosen arbitrarily, for every vertex  $v$  the polytope  $K$  is the convex hull of  $v$  and the facet not containing  $v$ . This implies that  $K$  is a simplex.  $\square$

## 4. PROOF OF THEOREM 1.2

Recall that a boundary point  $y \in \partial K$  is *strict* if it does not belong to any segment contained in  $\partial K$ . Note that points with positive Gaussian curvature and, more generally, regular exposed points are strict points (see [Sch] for the definitions). Clearly the boundary of a polytope does not contain any strict points, but there are other convex bodies having this property (for example, a cylinder).

As before it is enough to prove Theorem 1.2 in the case of  $r = 2$ . It follows from the theorem below.

**Theorem 4.1.** *Let  $K$  be a convex body whose boundary contains at least one strict point. Then there exist convex bodies  $L$  and  $M$  such that*

$$(4.1) \quad V(L, M, K[n-2])V_n(K) > V(L, K[n-1])V(M, K[n-1]).$$

*Proof.* First let us fix some notation. For  $a > 0$  and  $u \in \mathbb{S}^{n-1}$ , define the closed half-spaces:

$$H_a^+(u) = \{x \in \mathbb{R}^n : \langle x, u \rangle \geq a\} \quad \text{and} \quad H_a^-(u) = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq a\}.$$

Also set  $H_a(u) := H_a^+(u) \cap H_a^-(u)$ . With this notation, the supporting hyperplane of  $K$  whose unit normal vector is  $u$ , can be written as  $H_{h_K(u)}(u)$ .

Let  $y$  be a strict point of  $\partial K$  and  $u$  be a normal vector of  $K$  at  $y$ . Choose  $v \in \mathbb{S}^{n-1}$ , such that  $y|v^\perp \in \text{relint}(K|v^\perp)$ , where  $\text{relint}(K|v^\perp)$  denotes the relative interior of the body  $K|v^\perp$  in  $v^\perp$ . We claim that there exists  $\varepsilon > 0$ , such that

$$(4.2) \quad (K \cap H_{h_K(u)-\varepsilon}^-(u))|v^\perp = K|v^\perp.$$

To see this, assume that (4.2) is not true for all  $\varepsilon > 0$ . This means that for any  $\varepsilon > 0$ , there exists a point  $x_\varepsilon \in \partial K$ , such that  $x_\varepsilon|v^\perp \in \partial(K|v^\perp)$  and  $x_\varepsilon \in H_{h_K(u)-\varepsilon}^+(u)$ . Let  $x_0$  be an accumulation point of the set  $\{x_\varepsilon : \varepsilon > 0\}$ . Then, by compactness,  $x_0 \in \partial K$ ,  $x_0|v^\perp \in \partial(K|v^\perp)$ , and  $x_0 \in H_{h_K(u)}(u)$  (because  $x_0 \in H_{h_K(u)}^+(u)$  and  $x_0 \in K$ ). Note that, since  $x_0|v^\perp \in \partial(K|v^\perp)$  and  $y|v^\perp \in \text{relint}(K|v^\perp)$ , we have  $x_0 \neq y$ . It follows that the segment  $[x_0, y]$  is contained in a supporting hyperplane of  $K$ , thus  $[x_0, y] \subseteq \partial K$ , which contradicts the assumption that  $y$  is strict. Hence, (4.2) holds for some  $\varepsilon > 0$ .

Next, set  $K_\varepsilon := K \cap H_{h_K(u)-\varepsilon}^-(u)$ . Clearly,  $h_{K_\varepsilon} \leq h_K$ . We claim that there exists an open subset  $\beta \subset \partial K \setminus \partial K_\varepsilon$ , such that  $y \in \beta$  and

$$(4.3) \quad h_{K_\varepsilon}(u) < h_K(u), \quad \text{for all } u \in \sigma(K, \beta).$$

Suppose not. Then for any  $\delta$ -neighborhood  $\beta_\delta = (\partial K \setminus \partial K_\varepsilon) \cap B(y, \delta)$  of  $y$  there exists a unit vector  $u_\delta \in \sigma(K, \beta_\delta)$  such that  $h_K(u_\delta) = h_{K_\varepsilon}(u_\delta)$ . In other words, there exist points  $y_\delta \in \beta_\delta$  and  $x_\delta \in \partial K_\varepsilon$  lying in the same hyperplane  $H_K(u_\delta)$ . But then, by compactness, there exist a point  $x \in \partial K_\varepsilon$  and a unit vector  $u$ , which is normal for  $K$  at  $y$  and at  $x$ . This shows again that the points  $y$  and  $x$  of  $K$  lie in the same supporting hyperplane  $H_K(u)$ , thus  $[y, x]$  is a boundary segment of  $K$ , which contradicts our assumption. Therefore, (4.3) holds for some open set  $\beta \subseteq \partial K \setminus \partial K_\varepsilon$ .

Note, furthermore, that  $\tau(K, \sigma(K, \beta)) \supseteq \beta$ , thus  $\mathcal{H}^{n-1}(\tau(K, \sigma(K, \beta))) > 0$ , which shows that

$$(4.4) \quad S(K, \sigma(K, \beta)) > 0.$$

Now we are ready to exhibit examples of compact convex sets  $L$  and  $M$  satisfying (4.1). Set  $L = [-v, v]$  and  $M = K_\varepsilon$ . Then, by (5.3.23) in [Sch, p. 294] and applying (4.2) we obtain

$$V(L, M, K[n-2]) = V(K_\varepsilon|v^\perp, K|v^\perp[n-2]) = V_{n-1}(K|v^\perp) = V(L, K[n-1]).$$

On the other hand, by (4.3) and (4.4), we have:

$$\begin{aligned} V(M, K[n-1]) = V(K_\varepsilon, K[n-1]) &= \frac{1}{n} \int_{S^{n-1}} h_{K_\varepsilon} dS(K, \cdot) \\ &< \frac{1}{n} \int_{S^{n-1}} h_K dS(K, \cdot) = V_n(K). \end{aligned}$$

This shows that

$$V(L, M, K[n-2])V_n(K) > V(L, K[n-1])V(M, K[n-1]),$$

as asserted.  $\square$

**Remark 4.2.** One might ask the following: If  $K$  is a convex body whose boundary contains at least one strict point  $x$ , is it true that  $\partial K$  has an open neighborhood that does not contain any line segments, i.e.  $K$  is strictly convex in a neighborhood of  $x$ ? If yes, this would simplify the proof of Theorem 4.1 considerably. The following simple 3-dimensional example shows, however, that this is not the case. Take  $K$  equal to

$$\{x \in \mathbb{R}^3 : x_3 \leq 1\} \cap \text{conv}\left(\{(0, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_2^2\} \cup \{(x_1, 0, x_3) \in \mathbb{R}^3 : x_3 = x_1^2\}\right).$$

Then the origin is a strict point of the boundary of  $K$ , but no neighborhood of the origin is strictly convex.

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